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FORMULATION OF PROBLEMS OF FINDING STRESS FUNCTIONS WITH THE AID OF BIHARMONIC POTENTIALS

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Yu.D.Kopeykin (L'vov)

A general solution for the equation of statics of an elastic body in the presence of volume forces, in the form of biharmonic volume potentials, is derived. Discontinuity formulas for the limiting values of derivatives of volume potentials for single and double layers, at various points of the body surface are presented, based on the fundamental solution of the biharmonic equation $\xi = r/2$.

It is known that the general solution of the equations of statics of an elastic body in the absence of volume forces can be represented in the form of three harmonic and one biharmonic stress functions. The article first shows how this general solution can be obtained in the presence of volume forces; the particular solution is then derived in the form of so-called biharmonic volume potentials.

For a general solution of the basic boundary problems, the use of the derivatives of the biharmonic potentials for single and double layers is proposed. The article presents the "discontinuity formulas" for the limiting values of these derivatives at various points of the body surface.

The biharmonic potentials used in this work are based on the fundamental solution of the biharmonic equation $\xi = r/2$. However, other fundamental solutions may also be taken for this purpose.

To define the concept of biharmonic potentials, introduced by the present writer (Bibl.1, 2), let us apply the Gauss-Ostrogradskiy theorem to the sum of integrals

$$\sum_{i=1}^{3}\sum_{j=1}^{3}\int_T \zeta \frac{\partial^4\xi}{\partial x_i^2 \partial x_j^2}\, dt$$

where T is a domain of three-dimensional space; x_i, i = 1, 2, 3 are rectilinear orthogonal coordinates; $dt = dx_1dx_2dx_3$ is an element of the domain T; ζ, ξ are quadruply differentiable functions.

We thus have (Bibl.1)

$$\int_T (\zeta\nabla^4\xi - \xi\nabla^4\zeta)\, dt = \int_S \left[\zeta\frac{d}{dn}(\nabla^2\xi) - (\nabla^2\xi)\frac{d\zeta}{dn} + (\nabla^2\zeta)\frac{d\xi}{dn} - \xi\frac{d}{dn}(\nabla^2\zeta)\right] ds,$$

* Numbers in the margin indicate pagination in the original foreign text.

where S is the smooth Lyapunov boundary of the domain T; ds is an element of the boundary S; n is the direction of the outer normal with respect to S.

Let us assume that the function ζ is the fundamental solution of the biharmonic equation (Bibl.2), i.e.,

$$\zeta = \frac{1}{2} r, \tag{1}$$

where r is the distance between the variable points P_0 and P of the domain T + S.

The point P, over whose x_i coordinates the integration is carried out, is termed a current point, while the point P_0 with the coordinates x_{i0} is termed a fixed point.

Differentiating with respect to the coordinates of the point P, we have

$$\frac{d\zeta}{dn} = \frac{1}{2}\cos\varphi; \qquad \nabla^2\zeta = \frac{1}{r}; \qquad \frac{d}{dn}(\nabla^2\zeta) = -\frac{\cos\varphi}{r^2}, \tag{2}$$

where φ is the angle between the normal n, plotted at the point P, and the direction of the vector $\bar{r} = \overrightarrow{P_0P}$; α_i ; i = 1, 2, 3 are the direction cosines of the normal n; β_i , i = 1, 2, 3 are the direction cosines of the vector $\bar{r}$; /105

$$\cos\varphi = \sum_{i=1}^{3} \alpha_i\beta_i.$$

Following the substitution [cf. (Bibl.2)] of eqs.(1, 2), the integral formula given above is written as

$$4\pi\xi = \int_S \left[\frac{1}{r}\underline{\frac{d\xi}{dn}} + \underline{\xi}\frac{\cos\varphi}{r^2} + \frac{r}{2}\underline{\frac{d}{dn}(\nabla^2\xi)} - \underline{(\nabla^2\xi)}\frac{\cos\varphi}{2}\right] ds - \int_T \frac{r}{2}(\nabla^4\xi)\, dt. \tag{3}$$

On the left-hand side of eq.(3), the function ξ depends on the coordinates of the fixed point; on the right-hand side, it is regarded as a function of coordinates of the current point. The values of the functions at points inside the boundary S of the domain T are underlined. Equation (3) is analogous to the basic integral formula of the theory of harmonic functions (Green's function; Bibl.1).

The first two terms on the right-hand side of eq.(3) represent the Newtonian potentials of the single and double layers; their sum will be a harmonic function. The remaining three terms are proportional to the polar static moments of the single and double layers, as well as to the volume with respect to the fixed point. Hence the proposal (Bibl.2) that they be termed biharmonic potentials of the single layer, the double layer, and the volume. The polar static moment is construed as the integral of the product of the element and the distance r from the pole P_0.

The biharmonic potentials of the single and double layers satisfy the biharmonic equation.

Certain properties of biharmonic potentials were investigated elsewhere (Bibl.2).

Consider the properties of the third derivatives of the single-layer biharmonic potential and the second derivatives of the double-layer biharmonic potential, on examining their variation in the neighborhood of points at the surface S.

Let g be the biharmonic potential of a single layer, with the density ν of surface distribution

$$g = \frac{1}{2} \int_S \nu r ds. \tag{4}$$

Laying the outer normal n_0 through a point A of the surface S, we mark on it the fixed point P_0.

We then calculate the third-order derivatives of g with respect to the coordinates of the fixed point and the direction of the normal n_0

$$\frac{\partial^3 g}{\partial x_{i0}^2 \partial n_0} = \frac{1}{2} \int_S \frac{\nu}{r^2} [2\alpha_{i0}\beta_i + (1 - 3\beta_i^2)\cos\psi]\, ds;$$

$$\frac{\partial^3 g}{\partial x_{i0} \partial x_{j0} \partial n_0} = \frac{1}{2} \int_S \frac{\nu}{r^2} (\alpha_{i0}\beta_j + \alpha_{j0}\beta_i - 3\beta_i\beta_j \cos\psi)\, ds; \quad i \neq j, \tag{5}$$

where α_{i0}, i = 1, 2, 3 are the direction cosines of the normal n_0; ψ is the angle made by the directions of the normal n_0 and the vector $\bar{r}$: /106

$$\cos\psi = \sum_{i=1}^{3} \alpha_{i0}\beta_i.$$

We calculate the limiting values of the derivatives (5) when the fixed point P_0 tends toward the point A on the surface S. We decompose each of the integrals (5) into two additive integrals, one of which is taken over the neighborhood Δ of the point A and the other, over the remainder of the surface S. On determining the first addend, we equate to zero the distance between the points P_0 and A and then also the dimension of the neighborhood Δ. On passing to the limit, we have

$$\underline{\frac{\partial^3 g}{\partial x_{i0}^2 \partial n_0}} = \frac{2\pi}{\eta} \alpha_{i0}^2 \nu_0 + \frac{1}{2} \int_S \frac{\nu}{r^2} [2\alpha_{i0}\beta_i + (1 - 3\beta_i^2)\cos\psi]\, ds; \tag{6}$$

$$\frac{\partial^3 g}{\partial x_{i0}\partial x_{j0}\partial n_0} = \frac{2\pi}{\eta}\alpha_{i0}\alpha_{j0}\nu_0 + \frac{1}{2}\int\limits_S \frac{\nu}{r^2}(\alpha_{i0}\beta_j + \alpha_{j0}\beta_i - 3\beta_i\beta_j\cos\psi)\,ds; \quad i \neq j.$$

where ν_0 is the density ν at the limiting point; $\eta = 1$ if P_0 tends toward A inside the surface S; $\eta = -1$ if P_0 tends toward A outside the surface S.

The integrals in eqs.(6) denote the direct values of the derivatives at the point A. V.D.Kupradze (Bibl.3) showed that singular integrals of this type converge in the presence of a limited density ν.

Consider the biharmonic potential h of a double layer of density $\varkappa$ on the surface S

$$h = \frac{1}{2}\int\limits_S \varkappa\cos\varphi\,ds. \tag{7}$$

Let us then calculate the second derivatives of h with respect to the coordinates of the fixed point. We find the limiting values of these derivatives at the point A of the surface S, proceeding in the same manner as in deriving eqs.(6). This yields

$$\frac{\partial^2 h}{\partial x_{i0}^2} = -\frac{2\pi}{\eta}\alpha_{i0}^2\varkappa_0 - \frac{1}{2}\int\limits_S \frac{\varkappa}{r^2}[2\alpha_i\beta_i + (1 - 3\beta_i^2)\cos\varphi]\,ds;$$

$$\frac{\partial^2 h}{\partial x_{i0}\partial x_{j0}} = -\frac{2\pi}{\eta}\alpha_{i0}\alpha_{j0}\varkappa_0 - \frac{1}{2}\int\limits_S \frac{\varkappa}{r^2}(\alpha_i\beta_j + \alpha_j\beta_i - 3\beta_i\beta_j\cos\varphi)\,ds; \quad i \neq j. \tag{8}$$

where $\varkappa_0$ is the value of the density $\varkappa$ at the limiting point.

The meaning of the coefficient η has been clarified above.

Let us now consider the application of biharmonic potentials to the solution of three-dimensional problems of the statics of elastic bodies. We write the equation of elasticity

$$\nabla^2\bar{V} + \frac{1}{1-2\mu}\overline{\mathrm{grad}}\,\mathrm{div}\,\bar{V} + \frac{1}{G}\bar{X} = 0. \tag{9}$$

where $\bar{V}$ is the vector of elastic displacement; $\bar{X}$ is the vector of volume force with the components X_1, X_2, X_3; μ is Poisson's ratio; and G is the modulus of elasticity in compression.

Let us replace the function div $\bar{V}$ in eq.(9) and assume that this function is proportional to the Laplacian of some other function ξ_1 /107

$$\mathrm{div}\,\bar{V} = (1-2\mu)\nabla^2\xi_1. \tag{10}$$

Then, eq.(9) becomes a Poisson equation

$$\nabla^2(\bar{V} + \overline{\mathrm{grad}}\,\xi_1) = -\frac{\bar{X}}{G}. \tag{11}$$

The solution of eq.(11) represents the sum of the harmonic vector $\bar{a}$ and the Newtonian vector potential of volume T with the volume density $\bar{X}/G$. Hence,

$$\bar{V} = \bar{a} - \overline{\mathrm{grad}}\,\xi_1 + \frac{1}{4\pi G}\int_T \bar{X}\frac{1}{r}\,dt. \tag{12}$$

Then,

$$\mathrm{div}\,\bar{V} = \mathrm{div}\,\bar{a} - \nabla^2\xi_1 + \frac{1}{4\pi G}\int_T \bar{X}\,\overline{\mathrm{grad}}\,\frac{1}{r}\,dt = \mathrm{div}\,\bar{a} - \nabla^2\xi_1 +$$

$$+\frac{1}{4\pi G}\int_T X\frac{d}{dl}\left(\frac{1}{r}\right)dt = \mathrm{div}\,\bar{a} - \nabla^2\xi_1 + \frac{1}{4\pi G}\int_T X\nabla^2\left(\frac{d\,\frac{r}{2}}{dl}\right)dt,$$

where X is the modulus of the vector $\bar{X}$; l is the direction of the vector $\bar{X}$.

Comparing this last expression with eq.(10), we have

$$2(1-\mu)\nabla^2\xi_1 = \mathrm{div}\,\bar{a} + \frac{1}{4\pi G}\int_T X\nabla^2\left(\frac{d\,\frac{r}{2}}{dl}\right)dt. \tag{13}$$

We perform Laplace's operation with respect to both parts of eq.(13)

$$2(1-\mu)\nabla^4\xi_1 = \frac{1}{8\pi G}\int_T X\nabla^4\left(\frac{dr}{dl}\right)dt = \frac{1}{4\pi G}\nabla^4\frac{d}{dl}\int_T X\frac{r}{2}\,dt. \tag{14}$$

In accordance with the integral equation (3), the solution of eq.(14) will be written as the sum of a biharmonic function ξ and the l-direction derivative of the biharmonic potential of volume T with the volume density $X/8\pi G(1-\mu)$

$$\xi_1 = \xi + \frac{1}{8\pi G(1-\mu)}\frac{d}{dl}\int_T X\frac{r}{2}\,dt.$$

Then, eq.(12) will become

$$2(1-\mu)\nabla^2\xi = \mathrm{div}\,\bar{a}. \tag{15}$$

The vector of elastic displacement is written as

$$\bar{V} = \bar{a} - \overline{\text{grad}}\,\xi + \frac{1}{4\pi G} \int_T \left[\bar{X} \frac{1}{r} - \frac{X}{4(1-\mu)} \overline{\text{grad}} \frac{dr}{dl} \right] dt. \qquad (16)$$

The elastic displacement (16) is represented by three harmonic functions a_1, a_2, a_3, by the vector components $\bar{a}$ connected by the biharmonic function ξ, and by the particular solution of the elasticity equation in the form of the Newtonian and biharmonic potentials of volume T. /108

On examining the first three-dimensional boundary-value problem of statics, we write the boundary condition

$$\underline{\bar{V}} = \bar{f}, \qquad (17)$$

where $\underline{\bar{V}}$ is the value assumed by the vector $\bar{V}$ at points of the surface S; $\bar{f}$ is the vector specified at points of the surface S.

Let us then substitute eq.(16) into the boundary condition (17)

$$\underline{\bar{a}} - \underline{\overline{\text{grad}}}\,\xi = \bar{F}. \qquad (18)$$

In eq.(18), the values assumed by the vectors at points of the surface S are underlined; $\bar{F}$ is a known vector, with the components

$$F_i = f_i - \frac{1}{4\pi G} \int_T X \left[\frac{\gamma_i}{r} - \frac{1}{4(1-\mu)} \frac{\partial^2 r}{\partial x_{i0} \partial l} \right] dt, \qquad (19)$$

where γ_i are the direction cosines of the volume force vector $\bar{X}$.

To formulate the boundary conditions for the second elastostatic boundary-value problem, let us first express the stress tensor components with the aid of eq.(16), Cauchy's equation, and Hooke's law. After transformation, we have

$$\sigma_{ij} = G\left(\frac{\partial a_i}{\partial x_{j0}} + \frac{\partial a_j}{\partial x_{i0}} - 2\frac{\partial^2 \xi}{\partial x_{i0} \partial x_{j0}} + 2\delta_{ij}\mu \nabla^2 \xi \right) +$$

$$+ \frac{1}{4\pi} \int_T X \left(\gamma_i \frac{\partial \frac{1}{r}}{\partial x_{j0}} + \gamma_j \frac{\partial \frac{1}{r}}{\partial x_{i0}} - \frac{1}{2(1-\mu)} \frac{\partial^3 r}{\partial x_{i0} \partial x_{j0} \partial l} + 4\delta_{ij}\mu \frac{d \frac{1}{r}}{dl} \right) dt, \qquad (20)$$

where $\delta_{ij} = 1$ when $i = j$, and $\delta_{ij} = 0$ when $i \neq j$.

Let us then write the boundary conditions for the second boundary-value problem

$$\sum_{j=1}^{3} \underline{\sigma_{ij}} \alpha_{j0} = p_i; \quad i = 1, 2, 3, \qquad (21)$$

where p_i' are the components of the specified load vector; $\underline{\sigma}_{ij}$ are the values of the stress tensor components at points of the body surface S. Let us substitute eq.(20) into the boundary conditions. We then obtain boundary conditions expressed in the form of the stress functions

$$\frac{\underline{da_i}}{\underline{dn_0}} + \sum_{j=1}^{3} \frac{\underline{\partial a_j}}{\underline{\partial x_{i0}}} \alpha_{j0} - 2 \frac{\underline{\partial^2 \xi}}{\underline{\partial x_{i0} \partial n_0}} + 2\mu\alpha_{i0} \underline{\nabla^2 \xi} = \frac{1}{G} p_i; \quad i = 1, 2, 3. \tag{22}$$

where

$$p_i = p_i' - \frac{1}{4\pi} \int_T X \left[\gamma_i \frac{d\frac{1}{r}}{dn_0} + \frac{\partial \frac{1}{r}}{\partial x_{i0}} \cos(n_0 l) - \right. \\ \left. - \frac{1}{2(1-\mu)} \left(\frac{\partial^3 r}{\partial x_{i0} \partial n_0 \partial l} - 2\mu\alpha_{i0} \frac{d\frac{1}{r}}{dl} \right) \right] dt. \tag{23}$$

In condition (22), the values assumed by the functions at the surface points S are underlined. /109

We now formulate the third boundary-value problem in the determination of stress functions. Let us assume that the ratio of stress to displacement at the surface S of the body is specified such that

$$\sum_{j=1}^{3} \underline{\sigma_{ij}} \alpha_{j0} + \omega_i \underline{V_i} = \Phi_i'; \quad i = 1, 2, 3, \tag{24}$$

where ω_i, Φ_i are functions specified at the surface S.

Let us substitute eqs.(16), (20) into the boundary conditions (24). Then, we obtain, for the stress functions,

$$\frac{\underline{da_i}}{\underline{dn_0}} + \sum_{j=1}^{3} \frac{\underline{\partial a_j}}{\underline{\partial x_{i0}}} \alpha_{j0} - 2 \frac{\underline{\partial^2 \xi}}{\underline{\partial x_{i0} \partial n_0}} + 2\mu\alpha_{i0} \underline{\nabla^2 \xi} + \omega_i \left(\underline{a_i} - \frac{\underline{\partial \xi}}{\underline{\partial x_{i0}}} \right) = \Phi_i, \tag{25}$$

where Φ_i are components of the known vector determined at the surface points S according to the formula

$$\Phi_i = \Phi_i' - \frac{1}{4\pi} \int_T X \left\{ \frac{\omega_i(P_0)}{G} \left[\frac{\gamma_i}{r} - \frac{1}{4(1-\mu)} \frac{\partial^2 r}{\partial x_{i0} \partial l} \right] + \right. \\ \left. + \left[\gamma_i \frac{d\frac{1}{r}}{\partial n_0} + \frac{\partial \frac{1}{r}}{\partial x_{i0}} \cos(n_0 l) - \frac{1}{2(1-\mu)} \left(\frac{\partial^3 r}{\partial x_{i0} \partial n_0 \partial l} - 2\mu\alpha_{i0} \frac{d\frac{1}{r}}{dl} \right) \right] \right\} dt. \tag{26}$$

The fourth boundary-value problem, where stresses are specified over a part S_1 of the body surface S while displacements are specified over the remaining part S_2 of the surface, has boundary conditions of the form of eq.(22) for points of the sector S_1 and conditions of the form of eq.(18) for points of the sector S_2.

Thus, the boundary conditions for the four basic problems of statics, pertaining to the determination of stress functions, have been formulated with the aid of eqs.(18), (22), and (25). The effect of the volume force is expressed by eqs.(19), (23), (26), where the volume force serves as the density of the biharmonic and Newtonian volume potentials.

Thus, limiting formulas for the derivatives of single- and double-layer biharmonic potentials, needed for the further solution of boundary-value problems, have been obtained.

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